JOURNAL OF COMPUTATIONAL PHYSICS 46, 390–396 (1982)

Numerical Estimates of Hausdorff Dimension*

ALEXANDRE JOEL CHORIN

Department of Mathematics,
University of California, Berkeley, California 94720

Received December 2, 1981

Numerical methods for estimating Hausdorff dimension, useful in the analysis of turbulence, are explained and applied to a specific example. In particular, methods involving rescaling and approximation by Cantor sets are discussed.

1. INTRODUCTION

The need to estimate Hausdorff dimension numerically has arisen in several problems connected with turbulence theory, in particular in the analysis of three-dimensional vortex motion [2, 3], in the analysis of stochasticity in dynamical systems [1, 6, 13], and in the theory of turbulent flames [4]. Related notions of dimension are also of significance [6, 13], and other applications are likely to appear (see, e.g., [11]). The methods used in the literature to estimate Hausdorff dimension include a straightforward application of the definition [2], or of a modified definition [6, 13], and a rescaling technique followed by an approximation by Cantor sets [3]. The purpose of the present paper is to explain these methods and validate them by applying them to a set whose dimension is known.

The rescaling technique is of particular interest, because it strongly resembles the renormalization group techniques which are of increasing interest in physics (see, e.g., [15]); in the particular problem considered in this paper, the technique can be readily seen to be valid.

We begin by defining Hausdorff dimension [5]. Consider a compact set C; cover it by balls of radii $\rho_i \leqslant \rho$. Form the sum

$$S(D) = \sum \rho_i^D,$$

where D is a positive number. Consider the quantity

$$h(D) = \lim_{\rho \to 0} \liminf S(D), \tag{1}$$

* This work was supported in part by the Director, Office of Basic Energy Sciences, Engineering, Mathematical and Geosciences of the U. S. Department of Energy, under Contract W-7404-Eng-48. The work was performed while the author was a visiting member of the Courant Institute of Mathematical Sciences, whose hospitality is gratefully acknowledged.

0021-9991/82/060390–07$02.00/0

where the lim inf is taken over all covers with $\rho_i \leqslant \rho$; $h(D)$ is the measure of C in dimension D. The number

$$D^* = \text{greatest lower bound of } D \text{ for which } h(D) = \infty$$
$$= \text{least upper bound of } D \text{ for which } h(D) = 0$$

exists, and is the Hausdorff dimension of C. For a cube, $D^* = 3$; for a square, $D^* = 2$; for a segment, $D^* = 1$; according to [2, 3], the essential L_2 support of the vorticity in incompressible inviscid flow has dimension $D^* \sim 2.5$.

We shall apply our methods of estimation to the set Z of zeros of Brownian motion. Let $x(t)$, $0 \leqslant t \leqslant 1$, be a realization of Brownian motion (for a definition, see, e.g., [9]). The set of its zeros is the set of t's such that $x(t) = 0$. This set has, with probability 1, Hausdorff dimension $D^* = \frac{1}{2}$ [8, 14]. We shall use the following properties of Brownian motion: (i) The interpolation property [10]: If $x(\cdot)$ is normalized so that $x(1)$ has variance $\frac{1}{2}$, and if $x(t_1)$, $x(t_2)$ are known, then the conditional distribution of $x(t)$, $t_1 \leqslant t \leqslant t_2$, is given by

$$x(t) = x(t_1) + (t - t_1)(x(t_2) - x(t_1)/(t_2 - t_1) + ((t_2 - t)(t - t_1)/(t_2 - t_1))^{1/2} w, \tag{2}$$

where w is a Gaussian random variable with mean zero and variance $\frac{1}{2}$. (ii) Self similarity: $x(t)$, $0 \leqslant t \leqslant 1$, and $\sqrt{\alpha}\, x(t/\alpha)$, $\alpha > 0$, $0 \leqslant t \leqslant 1$, are equivalent processes.

Z is a random set. A comparison of our calculations here with the calculations in [2, 3] shows that Z is a much more irregular set than the sets of noninteger Hausdorff dimension generated by smooth differential equations, and the numerical estimation of the dimension of Z is not trivial. The methods used here are similar, but not identical, to the methods used in [2, 3]. Dimension is but one property of a set, and thought is required in each special case.

2. A Covering Scheme

We begin by covering the set of zeros of $x(\cdot)$ by segments of equal lengths (somewhat in the manner of the construction in [13]). Pick $x(1)$ by sampling the appropriate Gaussian distribution (an algorithm for sampling Gaussian distributions can be found, e.g., in [12]). By definition, $x(0) = 0$, and thus $[0, 1]$ contains at least one member of Z.

Divide $[0, 1]$ into $[0, \frac{1}{2}]$, $[\frac{1}{2}, 0]$. A value of $x(\frac{1}{2})$ can be found using (2): $x(\frac{1}{2}) = x(1)/2 + 2^{-3/2} w$. If $x(1/2)\, x(1) < 0$, $[1/2, 1]$ contains at least one zero.

More generally, define an iteration to be the following sequence of operations: Divide each one of the intervals from the preceding iteration into two halves; values of $x(\cdot)$ at the end points of the new intervals are either available from the preceding iteration or can be sampled by applying formula (2), which in this special case reads

$$x\,(\text{middle}) = \tfrac{1}{2}(x_- + x_+) + \tfrac{1}{2}\sqrt{\Delta}\, w, \tag{3}$$

where x (middle) is the value of $x(\cdot)$ in the middle of an old interval, x_- and x_+ are the values of $x(\cdot)$ at the left and right ends of an old interval, Δ is the length of a new interval, and w is defined as in (2).

Let x_-, x_+ now denote the values of $x(\cdot)$ at the ends of a new interval. If $x_- x_+ < 0$, the interval surely contains a zero of $x(\cdot)$. If $x_- x_+ > 0$, $|x_-| \geqslant K\sqrt{\Delta}$, $|x_+| \geqslant K\sqrt{\Delta}$, K large enough, there is a negligible probability that the interval contains a zero (see [10]) and the interval can be removed from further consideration. If neither of these conditions holds, the interval may or may not contain a zero.

We consider the quantity $\lim_{\Delta \to 0} n\Delta^D$, where $\Delta = 2^{-i}$ is the length of the intervals after the ith iteration, and n is the number of intervals which are known to contain zeros (they satisfy $x_- x_+ < 0$). The difference between this quantity and the quantity $h(D)$ in (1) lies in the fact that all the radii $\Delta/2$ are equal and also in the fact that the lim inf operation in (1) has not been carried out. Thus $\lim 2^{-D} n\Delta^D$ is an upper bound on $h(D)$. The number

$$\tilde{D} = \text{greatest lower bound of } D \text{ for which } n\Delta^D \to 0,$$

$$= \text{least upper bound of } D \text{ for which } n\Delta^D \to \infty,$$

is an upper bound on the dimension D^* of Z. We shall be computing $\tilde{D}$, and we shall assume without proof that $\tilde{D} = D^*$. From the numerical analysis point of view it does not matter whether this (highly plausible) assumption is correct, since we shall obtain estimates of $\tilde{D}$ with error estimates. In the calculations of [2, 3] the lim inf in (1) is computed correctly. In [13], a quantity analogous to $\tilde{D}$ is evaluated.

In Table I we display values of $n\Delta^D$ for several values of D as functions of the iteration i. For $D < D^*$, $n\Delta^D$ should be increasing; for $D > D^*$, $n\Delta^D$ should be

TABLE I

Values of $n\Delta^D$ as Functions of i and D

i	D							
	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70
1	0.35	0.24	0.17	0.12	0.088	0.062	0.044	0.031
2	0.34	0.23	0.16	0.11	0.075	0.051	0.035	0.024
3	0.27	0.17	0.11	0.078	0.051	0.034	0.022	0.014
4	0.51	0.32	0.20	0.13	0.084	0.053	0.034	0.021
5	0.46	0.28	0.17	0.10	0.067	0.041	0.025	0.015
6	0.68	0.40	0.24	0.14	0.085	0.050	0.030	0.017
7	0.74	0.42	0.29	0.14	0.080	0.046	0.026	0.015
8	0.97	0.53	0.29	0.16	0.091	0.051	0.028	0.015
9	1.04	0.55	0.29	0.16	0.085	0.045	0.024	0.015
10	1.06	0.55	0.28	0.14	0.079	0.039	0.020	0.010

decreasing. We set $K = 4$ (we shall show below that this is a large enough value of K). The calculation must be stopped after a finite number of iterations because the number of intervals quickly overwhelms the available computer memory and because Δ shrinks to below the underflow limit of the computer arithmetic. A reasonable person looking at Table I could conclude that D^* lies somewhere between 0.45 and 0.60—not a dramatically accurate answer. In the next section we shall see how this estimate can be improved without catastrophic expense.

3. RESCALING

Suppose the number of segments in the calculation (both the segments known to contain zeros and those which may turn out to contain zeros) exceeds a preset number n_0 (we usually set $n_0 = 100$). Then rescale n, i.e., throw away half the segments in such a way that the expected number of segments which contain zeros in the rejected half equals the number of such segments in the retained half. We shall call such a rejection "unbiased." An unbiased rejection can be easily achieved; for example, each time a segment is halved, store the parameters relating to one-half of the old segment in the array location in which the old segment was remembered and store the prameters which describe the other half at the end of the array. A rejection of the first or the second half of the resulting array is unbiased. Perform this rescaling at each iteration if needed. Segments which are not likely to contain zeros may, in the interest of efficiency, be thrown out as soon as they are generated.

If the segment length Δ become smaller than a preset Δ_0 (we usually set $\Delta_0 = 0.01$ or $\Delta_0 = 0.001$), rescale Δ, i.e., double Δ, and, in order to leave Z invariant, multiply all values x_-, x_+, by $\sqrt{2}$ (see the self similarity property (ii) above).

If there are n segments with a sure zero in the computer at the end of an iteration, there would have been $\sim Nn$ segments in the computer if rescaling had not been used, $N = 2^{l_1}$, where l_1 is the number of rescalings of n. If Δ is the length of the segments as stored in the computer, the real length is Δ/M, $M = 2^{l_2}$, where l_2 is the number of rescalings of Δ. Thus $D < D^*$ if $NM^{-D}n\Delta^D \to \infty$, $D > D^*$ if $NM^{-D}n\Delta^D \to 0$. We could again estimate D^* by following trends in the evolution of $NM^{-D}n\Delta^D$ for several values of D.

A sharper estimate is available, however. By construction, $1 \leqslant n \leqslant 2n_0$ and $\Delta_0 \leqslant \Delta \leqslant 2\Delta_0$. Thus $n\Delta^D$ is, for each D, a positive quantity bounded from above and bounded away from zero. Thus $NM^{-D}n\Delta^D$ tends to zero or infinity if NM^{-D} does. Therefore, as $N \to \infty$ and $M \to \infty$, we must have $M^{-D^*} = O(N)$ and

TABLE II

Estimates of D^* with 60 Iterations

0.49	0.54	0.51	0.50	0.54
0.46	0.53	0.45	0.52	0.48

TABLE III

Estimates of D^* with 600 Iterations

0.506	0.500	0.503	0.496

$D^* \to \log N/\log M$. In Table II we display values of $\log N/\log M$ obtained after 60 iterations in different runs, each run using a different sequence of random numbers to generate Z. In 60 iterations n is typically rescaled 6 to 7 times and Δ about 40 times. In Table III we display estimates of D^* obtained in a similar fashion with 600 iterations ($n_0 = 100$). These iterations are very inexpensive (each one takes a second or two on a VAX computer); after 600 iterations, the error in the estimate of D^* is under 1%; without rescaling, the same accuracy would have presumably required about $2^{400} \cong 10^{100}$ segments stored in the computer— an unimaginably expensive enterprise.

Finally, we can check how large K must be before we are reasonably sure that an interval contains no zeros. Numerical experiment shows that the estimates of D^* are independent of K as long as $K \geqslant 2$.

4. Approximation by Cantor Sets

We now describe a method for estimating the dimension of a set by approximating the given set by a suitable Cantor set.

The Cantor sets we shall use are constructed as follows: Consider the interval $[0, 1]$, and divide it into $m \geqslant 2$ segments of length $1/m$. Keep $s \leqslant m$ of these segments, $s > 1$, and throw out the other. Divide each one of the remaining segments into m pieces of equal lengths and throw out all but s these, etc. The remaining set has Hausdorff dimension $D^* = \log s/\log m$. This can be easily seen if one assumes that the measure of the remainder in dimension D^* is positive: The Hausdorff measure of disjoint sets is additive, and if a set B is similar to a set A with a similarity ratio L, the ratio of their measure in dimension D is L^D. Thus, if $h(D^*)$ is the measure of the remainder set in dimension D^*, we have

$$h(D^*) = (s/m^{D^*}))\, h(D^*),$$

i.e.,

$$D^* = \log s/\log m.$$

For details, see [7, 11].

Consider again the set Z. Let l_1 be an integer. Divide $[0, 1]$ into l_1 segments of equal lengths. Find the values of a realization of $x(\cdot)$ at the end points of those intervals, using the interpolation formula (2). This is done with ease if $l_1 = 2^{l_2}$, l_2

integer, since then formula (3) can be used recursively. Delete the intervals which are not likely to contain zeros, i.e., those characterized by $x_- x_+ > 0$, $|x_-| \geqslant 2\sqrt{\Delta}$, $|x_+| \geqslant 2\sqrt{\Delta}$, where as before Δ is the length of the interval and x_-, x_+ the values of $x(\cdot)$ at its ends. Suppose there are $l_3^{(1)}$ intervals left. Define $\bar{D}_1 = \log l_3^{(1)}/\log l_1$.

Throw away all but l_4 of the remaining intervals. Divide each one into $l_1 = 2^{l_2}$ pieces, decide which pieces are unlikely to contain zeros and throw them out. Suppose there are $l_3^{(2)}$ pieces left; there are $l_3^{(2)}/l_4$ such pieces in each of the starting l_4 intervals. Let $\bar{D}_2 = \log(l_3^{(2)}/l_4)/\log l_1$.

Keep iterating in this manner: Start with l_4 pieces, divide each one into l_1 smaller pieces, compute the number $l_3^{(i)}$ of pieces left, and let $\bar{D}_i = \log(l_3^{(i)}/l_4)/\log l_1$. Whenever the intervals become too small for convenient computation, rescale them, i.e., multiply their lengths Δ by a suitable factor A and multiply the x_-, x_+ by $\sqrt{A}$, as was done above.

If all the $l_3^{(i)}$ were equal to a fixed integer l_3, independently of the choice of intervals to subdivide, all the $\bar{D}_i$ would be equal, and the set remaining after an infinite sequence of rejections would be a Cantor set of Hausdorff dimension $\bar{D} = \bar{D}_i$ for all i. This set would be larger than Z because at each step we keep intervals which may contain zeros, but in fact will turn out to contain none. Thus $\bar{D} \geqslant D$. If the $l_3^{(i)}$ are not equal, we can view the numbers $\bar{D}_i$ as estimates of $\bar{D}$. The first 10 estimates of $\bar{D}$ are listed in Table IV. The averages of the $\bar{D}_i$ after 50 steps, with $l_2 = 5$, $l_4 = 4$, in one particular run, was 0.56 with standard deviation 0.025. Thus 0.56 ± 0.025 is an estimate of an upper bound $\bar{D}$ of D^*.

We now produce a construction which will provide an estimate of a lower bound $\underline{D}$ of D^*. Proceed exactly as in the preceding construction but change the criteria for rejecting intervals. Retain each interval which surely contains a zero ($x_- x_+ < 0$). Reject each interval not likely to contain a zero (defined as in the preceding section). Consider the remaining segments whose fate is uncertain (and which were retained in the preceding construction). Given an interval of length Δ, with values x_-, x_+, of $x(\cdot)$ at its extremities, let $p(\Delta, x_-, x_+)$ be the probability that it contains a zero. Construct an algorithm which retains the interval with probability p (and rejects it with probability $(1-p)$). This is easily done: Let k be an integer. Divide Δ into k subintervals of lengths Δ/k, and use (2) again to construct a Brownian arc connecting $x_- x_+$. If the resulting arc crosses the t axis, we keep the segment, and if it does not, we reject the segment. For k large enough (in practice, $k \geqslant 64$), we are keeping the segment with approximately probability p.

As before, if the $l_3^{(i)}$ were equal we would have a set with Hausdorff dimension $\underline{D} = \underline{D}_i \equiv \log(l_3^{(i)}/l_4)/\log(l_1)$, where the $l_3^{(i)}$ are the numbers of segments retained

TABLE IV

First 10 Estimates of $\bar{D}$, $l_2 = 5$, $l_3 = 4$

0.35	0.74	0.16	0.69	0.68
0.74	0.38	0.52	0.35	0.61

TABLE V

First 10 Estimates of $\underline{D}$, $l_2 = 5, l_3 = 4$

0.38	0.51	0.36	0.56	0.55
0.48	0.40	0.44	0.29	0.64

according to the new criteria, while l_1, l_4 are defined as before. $\underline{D}$ is a lower bound on D^* because the remaining set is too small—each doubtful segment has a probability p of being rejected, but if it kept in one iteration it may be rejected later because the applications of the interpolation formula (2) are independent. The $\underline{D}_i$, however, are not equal and are only estimates of $\underline{D}$. In Table V we list the first 10 estimates of $\underline{D}_i$ of $\underline{D}$, with $l_2 = 5$, $l_4 = 4$; the average of the $\underline{D}_i$ after 50 steps was 0.48 ± 0.015. Thus,

$$\underline{D} \leqslant D^* \leqslant \bar{D},$$

and we have found

$$\bar{D} = 0.56 \pm 0.025$$

$$\underline{D} = 0.48 \pm 0.015.$$

$\underline{D} - \bar{D} \neq 0$ because Z is a random set, unlike the sets encountered in the applications to differential equations. We have obtained fairly sharp estimates of $\bar{D}$ and $\underline{D}$ but a rather poor estimate of D^*. The best estimate of D^* was obtained by the rescaling of the preceeding section.

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